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The various power decays of the survival probability at long times for a free quantum particle

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Abstract

The long time behaviour of the survival probability of the initial state and its dependence on the initial states are considered, for the one-dimensional free quantum particle. We derive the asymptotic expansion of the time evolution operator at long times, in terms of the integral operators. This enables us to obtain the asymptotic formula for the survival probability of the initial state $\psi(x)$, which is assumed to decrease sufficiently rapidly at large $|x|$. We then show that the behaviour of the survival probability at long times is determined by that of the initial state ψ at zero momentum $k = 0$. Indeed, it is proved that the survival probability can exhibit the various power decays such as t^{-2m-1} for an arbitrary non-negative integer m as $t \rightarrow \infty$, corresponding to the initial states with the condition $\hat{\psi}(k) = O(k^m)$ as $k \rightarrow 0$.

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1. Introduction

Decaying quantum systems such as an α -decaying nucleus are often described by the survival probability of the initial state, which is the probability of finding the initial state in the state at a later time. One of the remarkable properties of the survival probability is its power decay law at long times. This is a mathematically predicted nature for systems which possess a continuous energy spectrum bounded from below [1]. It is actually seen in many models (see e.g. [2–4] and the references therein). On the other hand, there still remains the difficulty in observing such power decays in real experiments [5–7]. Hence, further theoretical and experimental investigations of the power decay law are much required.

One of the fundamental and important models that exhibits the power decay law is the free particle system, from which we can gain many insights into the dynamics of the survival probability and also the spatial wave packet. Recently, another aspect of the power-law regime at long times is revealed for the one-dimensional free particle system, in connection with the

initial states. As we know, the spatial wave packets for this system are expected to decay like $t^{-1/2}$ at long times t . We are assured of such a decay for the Gaussian wave packet. However, it is not necessarily valid for an arbitrary initial state. In fact, if we take a spatial power-law wave packet as an initial state, the ‘anomalous decay’ of its maximum can occur with the form $t^{-\alpha/2}$ ($1/2 < \alpha < 1$) [8–10]. This is obviously slower than the well-known $t^{-1/2}$ decay. Faster decay than $t^{-1/2}$ is also studied for the initial wave packets which vanish at zero momentum, in association with the dwell time [11] and the time operator (see [12] and appendix A). The latter refers to the time evolution of the survival probability. Therefore, the asymptotic decay form of the wave packet for the free particle system depends on the initial states in a considerable way. However, it seems that we are still not getting a clear perspective of this new aspect of the power decay law. Our aim in the present paper is to find the strict condition of the initial states for various power decays in the one-dimensional free particle system, and to clarify the underlying mechanism for such decays. In particular, we restrict ourselves to the survival probability which will show various power decays similar to the spatial wave packet.

To this end, we introduce a systematic approach which consists of two procedures. We first derive in section 3 the asymptotic expansion of the time evolution operator $\exp(-itH_0)$ as $t \rightarrow \infty$, where H_0 is the free Hamiltonian for the one-dimensional free particle system. It is formally written as

$$\exp(-itH_0) = \pi^{-1} \sum_{j=0}^{\infty} (-1)^{j-1} \Gamma(j+1/2) (it)^{-j-1/2} G_{2j} \quad (1.1)$$

where $\Gamma(z)$ is the gamma function, and G_{2j} are some integral operators. This kind of asymptotic expansion has already been developed by Rauch [13], Jensen and Kato [14], and Murata [15] with detailed analyses (see also a recent comment by Amrein [16]). Their results concern the time evolution operator for systems with a short-range potential $V(x)$. The asymptotic expansion (1.1) for the free particle system is not evaluated by the authors mentioned above, however, it can be achieved without difficulty, following their technique for the potential system. The survival probability of the initial state ψ is defined by the square modulus of the survival amplitude of ψ , that is

$$\langle \psi, \exp(-itH_0)\psi \rangle.$$

We see from (1.1) that $\langle \psi, \exp(-itH_0)\psi \rangle = O(t^{-1/2})$ only if $\langle \psi, G_0\psi \rangle \neq 0$. In other words, if the following condition,

$$\langle \psi, G_{2j}\psi \rangle = 0 \quad j = 0, 1, \dots, m-1 \quad (1.2)$$

holds for some integer m , we can obtain another asymptotic decay form of $\langle \psi, \exp(-itH_0)\psi \rangle$, $t^{-m-1/2}$, faster than $t^{-1/2}$. Thus, our remaining procedure is to interpret condition (1.2) as the behaviour of the initial state ψ in, e.g., position or momentum space. This is achieved in section 4, and we will finally obtain a remarkable conclusion: if the initial state ψ behaves like $\hat{\psi}(k) = O(k^m)$ at zero momentum $k = 0$, then

$$|\langle \psi, \exp(-itH_0)\psi \rangle|^2 = \pi^{-1} t^{-2m-1} \Gamma(m+1/2)^2 |\langle \psi, G_{2m}\psi \rangle|^2 + o(t^{-2m-1}) \quad (1.3)$$

as $t \rightarrow \infty$, where ψ is assumed to decrease sufficiently rapidly at large $|x|$, but otherwise arbitrary. Hence, the behaviour of the initial state ψ at zero momentum, i.e. $\hat{\psi}(k) = O(k^m)$, completely characterizes the asymptotic decay form of the survival probability of ψ . This fact is also expected from the results on the decays faster than $t^{-1/2}$ for the one-dimensional free particle system [11, 12].

The paper is organized as follows. We first consider in section 2 the asymptotic behaviour of the free resolvent $(H_0 - z)^{-1}$ at small and large energies. This study is necessary for the proof

of theorem 3.1 in section 3, where the asymptotic expansion (1.1) is derived. The derivation essentially follows the method used by Jensen and Kato [14]. To derive the asymptotic formula (1.3) for the survival probability, it is enough to derive that for the survival amplitude. The latter is accomplished in theorem 4.3 in section 4. Concluding remarks are given in section 5.

2. The free resolvent in one dimension

We here define the free Hamiltonian in one dimension, $H_0 := P^2$, where P is the momentum operator defined as $P := -iD_x$, D_x is the differential operator on $L^2(\mathbb{R})$. Then, the free resolvent $R_0(z) := (H_0 - z)^{-1}$ is explicitly represented as an integral operator on $L^2(\mathbb{R})$ [17]

$$(R_0(z)\psi)(x) = -\frac{1}{2iz^{1/2}} \int_{\mathbb{R}} \exp(iz^{1/2}|x - y|)\psi(y) dy \tag{2.1}$$

for all $\psi \in L^2(\mathbb{R})$, where z belongs to $\mathbb{C}_+ := \{z \in \mathbb{C} | \text{Im } z > 0\}$, and $\text{Im } z^{1/2} > 0$. The resolvent $R_0(z)$ is analytic in z . We, however, intend to regard it as an operator which maps $L^{2,s}(\mathbb{R})$ to $L^{2,-s'}(\mathbb{R})$ for positive s and s' . Here $L^{2,s}(\mathbb{R})$, defined for an arbitrary real s , is the weighted L^2 -space with the norm

$$\|\psi\|_s := \left[\int_{\mathbb{R}} (1 + x^2)^s |\psi(x)|^2 dx \right]^{1/2}$$

where $\psi \in L^{2,s}(\mathbb{R})$ means $\|\psi\|_s < \infty$. For positive s and s' , the relation $L^{2,s}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset L^{2,-s'}(\mathbb{R})$ holds. In addition, we denote by $\mathbf{B}(s, -s')$ the Banach space of the bounded operators M from $L^{2,s}(\mathbb{R})$ to $L^{2,-s'}(\mathbb{R})$, with the norm

$$\|M\|_{s,-s'} := \sup_{\psi \in L^{2,s}(\mathbb{R}), \psi \neq 0} \frac{\|M\psi\|_{-s'}}{\|\psi\|_s}.$$

Note that $M \in \mathbf{B}(s, -s')$ means the finiteness of its norm $\|M\|_{s,-s'} < \infty$. The reason for this kind of preparation for the free resolvent will be clear in the last part of lemma 2.1.

Lemma 2.1. *For $s, s' > 1/2$, $R_0(z)$ belongs to $\mathbf{B}(s, -s')$, and is continuously extended to $\overline{\mathbb{C}_+} \setminus \{0\}$ where $\overline{\mathbb{C}_+}$ is the closure of \mathbb{C}_+ .*

Proof. We have an estimation

$$\begin{aligned} \|R_0(z)\psi\|_{-s'}^2 &= \int_{\mathbb{R}} (1 + x^2)^{-s'} \left| \frac{1}{2iz^{1/2}} \int_{\mathbb{R}} \exp(iz^{1/2}|x - y|)\psi(y) dy \right|^2 dx \\ &\leq \frac{\|\psi\|_s^2}{|2iz^{1/2}|^2} \int_{\mathbb{R}} (1 + x^2)^{-s'} dx \int_{\mathbb{R}} (1 + y^2)^{-s} dy < \infty \end{aligned}$$

for all $\psi \in L^{2,s}(\mathbb{R})$ and $z \in \mathbb{C}_+$. This result clearly holds for $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ and the last part of the statement is also proved. \square

We use the same symbol $R_0(z)$ for the extension of $R_0(z)$ to $\overline{\mathbb{C}_+} \setminus \{0\}$. The free resolvent $R_0(z)$ is formally expanded around $z = 0$,

$$R_0(z) = \sum_{j=0}^{\infty} (iz^{1/2})^{j-1} G_j \tag{2.2}$$

where G_j ($j = 0, 1, \dots$) is an integral operator acting on the suitable vectors ψ ,

$$(G_j\psi)(x) := -\frac{1}{2j!} \int_{\mathbb{R}} |x - y|^j \psi(y) dy. \tag{2.3}$$

Lemma 2.2. *The integral operator G_j is a Hilbert–Schmidt operator that belongs to $\mathbf{B}(s, -s')$ with $s, s' > j + 1/2$.*

Proof. Note that the statement in the lemma is equivalent to $(1+x^2)^{-s'/2}G_j(1+y^2)^{-s/2}$ being a Hilbert–Schmidt operator on $L^2(\mathbb{R})$. The latter is easily seen from the relation

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|x-y|^{2j}}{(1+x^2)^{s'}(1+y^2)^s} dx dy \leq 2^{2j} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|x|^{2j} + |y|^{2j}}{(1+x^2)^{s'}(1+y^2)^s} dx dy < \infty$$

for every $s, s' > j + 1/2$. \square

The validity of the formal expansion (2.2) is ensured at small energies, in the following sense.

Lemma 2.3. *Let $k = 0, 1, \dots$. If $R_0(z)$ is approximated by a finite series in (2.2) up to $j = k$, the remainder is $o(|z|^{(k-1)/2})$ as $|z| \rightarrow 0$, in the norm of $\mathbf{B}(s, -s')$ with $s, s' > k + 1/2$. In the same sense, (2.2) can be differentiated in $z \in \overline{\mathbb{C}_+} \setminus \{0\}$ any number of times for appropriate s and s' , that is the r th derivative in z of the approximating finite series is equal to $(d^r/dz^r)R_0(z)$ up to an error of $o(|z|^{(k-1)/2-r})$ in the norm of $\mathbf{B}(s, -s')$ with $s, s' > k + r + 1/2$.*

Proof. We first consider the case of $k = 0$. Suppose that r is a non-negative integer, $s, s' > r + 1/2$, and $\psi \in L^{2s}(\mathbb{R})$. Then it follows that

$$\begin{aligned} & \left\| \frac{d^r R_0(z)}{dz^r} \psi - \frac{d^r (iz^{1/2})^{-1}}{dz^r} G_0 \psi \right\|_{-s'}^2 \\ &= \int_{\mathbb{R}} (1+x^2)^{-s'} \left| \int_{\mathbb{R}} \frac{d^r}{dz^r} \left[\frac{1}{2iz^{1/2}} (\exp(iz^{1/2}|x-y|) - 1) \right] \psi(y) dy \right|^2 dx \\ &\leq \frac{\|\psi\|_s^2}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} (1+x^2)^{-s'} (1+y^2)^{-s} \left[A|z|^{-1-2r} |\exp(iz^{1/2}|x-y|) - 1|^2 \right. \\ &\quad \left. + \sum_{m,m'} A_{m,m'} |z|^m |x-y|^{m'} \right] dx dy \end{aligned} \quad (2.4)$$

where A and $A_{m,m'}$ are positive constants, and m and m' are integers, satisfying $-1-2r < m \leq -1-r$ and $0 < m' \leq 2r$, respectively. When $r = 0$, there is no contribution from the summation $\sum_{m,m'}$. By the dominated convergence theorem, we see that (2.4) divided by $|z|^{-1-2r}$ goes to 0 as $z \rightarrow 0$. In the same way, for $k = 1, 2, \dots$, we have

$$\begin{aligned} & \left\| \frac{d^r R_0(z)}{dz^r} \psi - \sum_{j=0}^k \frac{d^r (iz^{1/2})^{j-1}}{dz^r} G_j \psi \right\|_{-s'}^2 \\ &= \int_{\mathbb{R}} (1+x^2)^{-s'} \left| \int_{\mathbb{R}} \frac{d^r}{dz^r} \left[\frac{(iz^{1/2})^{k-1} |x-y|^k}{2(k-1)!} \right. \right. \\ &\quad \left. \left. \times \left(\int_0^1 t^{k-1} \exp(iz^{1/2}|x-y|(1-t)) dt - \frac{1}{k} \right) \right] \psi(y) dy \right|^2 dx \\ &\leq \frac{\|\psi\|_s^2}{|2(k-1)!|^2} \int_{\mathbb{R}} \int_{\mathbb{R}} (1+x^2)^{-s'} (1+y^2)^{-s} \left[B|z|^{k-1-2r} |x-y|^{2k} \right. \\ &\quad \left. \times \left| \int_0^1 t^{k-1} \exp(iz^{1/2}|x-y|(1-t)) dt - \frac{1}{k} \right|^2 + \sum_{m,m'} B_{m,m'} |z|^m |x-y|^{m'} \right] dx dy \end{aligned} \quad (2.5)$$

for $s, s' > k + r + 1/2$, where B and $B_{m,m'}$ are positive constants, and m and m' are integers, satisfying $k - 1 - 2r < m \leq k - 1 - r$ and $2k < m' \leq 2(k + r)$, respectively. Taking the limit $z \rightarrow 0$, one can see that (2.5) divided by $|z|^{k-1-2r}$ goes to 0. This completes the proof of the lemma. \square

On the other hand, we also have the following lemma with respect to the asymptotic behaviour of $R_0(z)$ at large energies.

Lemma 2.4. *Let $k = 0, 1, \dots$ and $s, s' > k + 1/2$. Then $R_0(z)$ is k -times differentiable in $z \in \overline{\mathbb{C}_+} \setminus \{0\}$, in $\mathbf{B}(s, -s')$, and it behaves like*

$$\frac{d^r R_0(z)}{dz^r} = O(|z|^{-(r+1)/2}) \quad r = 0, 1, \dots, k$$

as $|z| \rightarrow \infty$ in the norm of $\mathbf{B}(s, -s')$.

Proof. Suppose that $s, s' > k + 1/2$ and $\psi \in L^{2,s}(\mathbb{R})$, we have

$$\begin{aligned} \left\| \frac{d^r R_0(z)}{dz^r} \psi \right\|_{-s'}^2 &\leq \frac{\|\psi\|_s^2}{4} \int_{\mathbb{R}} dx \frac{1}{(1+x^2)^{s'}} \int_{\mathbb{R}} dy \frac{1}{(1+y^2)^s} \\ &\times \left[D|z|^{-1-r} |x-y|^{2r} + \sum_{m,m'} D_{m,m'} |z|^m |x-y|^{m'} \right] < \infty \end{aligned}$$

where D and $D_{m,m'}$ are positive constants, and m and m' are integers, satisfying $-1 - 2r \leq m < -1 - r$ and $0 \leq m' < 2r$, respectively. Then, the right-hand side is $O(|z|^{-1-r})$ as $|z| \rightarrow \infty$. \square

3. Asymptotic expansion of the time evolution operator

In order to derive the asymptotic expansion of $\exp(-itH_0)$ in (1.1), we first define the spectral density denoted by $E'(\lambda) := (2\pi i)^{-1}(R_0(\lambda) - \overline{R_0}(\lambda))$ for all $\lambda > 0$, where

$$(\overline{R_0}(\lambda)\psi)(x) := \frac{1}{2i\lambda^{1/2}} \int_{\mathbb{R}} dy \exp(-i\lambda^{1/2}|x-y|)\psi(y)$$

for every $\psi \in L^{2,s}(\mathbb{R})$ with $s > 1/2$. The operator $\overline{R_0}(\lambda)$ is considered as the limit of $R_0(\lambda + i\epsilon)$ in $\epsilon \uparrow 0$. $E'(\lambda)$ clearly belongs to $\mathbf{B}(s, -s')$ with $s, s' > 1/2$, and it has the same properties as $R_0(\lambda)$ described in lemmas 2.1, 2.3 and 2.4. Substituting expansion (2.2) and the corresponding one of $\overline{R_0}(\lambda)$ into $E'(\lambda)$, we have

$$E'(\lambda) = \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \lambda^{j-1/2} G_{2j} + F_n(\lambda) \tag{3.1}$$

where $F_n(\lambda)$ is the remainder. It should be noted that there are no integer powers in λ . We next focus our attention on the following formula,

$$\exp(-itH_0) = \lim_{R \rightarrow \infty} \lim_{r \downarrow 0} \int_r^R E'(\lambda) e^{-it\lambda} d\lambda \tag{3.2}$$

valid in $\mathbf{B}(s, -s')$ for $s, s' > 1/2$. The integration in the above can also be regarded as the complex integral of $(2\pi i)^{-1}R_0(z) e^{-itz}$ with the contour enclosing the spectrum of H_0 , i.e. $[0, \infty)$. This formula is shown in appendix B. Then, the asymptotic expansion of $\exp(-itH_0)$ at large t is obtained from formula (3.2) together with expansion (3.1). To be precise, we can show

Theorem 3.1. *Let $n = 0, 1, \dots$ and $s, s' > \max\{3n + 3/2, 5/2\}$. Then it follows that*

$$\exp(-itH_0) = \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2) (it)^{-j-1/2} G_{2j} + o(t^{-n-1/2}) \quad (3.3)$$

as $t \rightarrow \infty$, in the norm of $\mathbf{B}(s, -s')$.

Note that the asymptotic form of $\exp(-itH_0)$ at large times is only determined by the behaviour of the free resolvent $R_0(z)$ at small energies. Our proof follows the procedure proposed by Jensen and Kato [14], and is given in appendix C.

It is interesting to rewrite formula (3.3) by using the ‘generalized’ zero-energy eigenfunction of H_0 , i.e. $\varphi_0(x) := (2\pi)^{-1/2}$. Since $G_0 = -\pi \langle \varphi_0, \cdot \rangle \varphi_0$ from (2.3), we have an alternative expression of (3.3) as

$$\exp(-itH_0) = \pi^{1/2} (it)^{-1/2} \langle \varphi_0, \cdot \rangle \varphi_0 + o(t^{-1/2}) \quad t \rightarrow \infty. \quad (3.4)$$

Note that this has the same structure as the asymptotic expansions of the one- and three-dimensional systems with short-range potential V , which have no zero-energy eigenstate but zero-energy resonance. The existence of the former implies that there is a zero-energy eigenfunction belonging to the L^2 -space, while that of the latter corresponds to the situation in which there is a function ψ_0 , that is not in $L^2(\mathbb{R})$ but satisfies $G_0 V \psi_0 = 0$ for one dimension or $(1 + G_0^{(3)} V) \psi_0 = 0$ for three dimension. The system with the zero-energy resonance is known not to be ‘generic’ [13–16], and in such a case φ_0 in (3.4) is replaced by ψ_0 . In this sense, the one-dimensional free particle system is considered to be exceptional. In contrast, the three-dimensional free particle system seems to be generic, since the $t^{-1/2}$ -term does not appear in the expansion of $\exp(-itH_0)$. This is because the free resolvent for the three-dimensional case has no singularity at the origin, while it appears in (2.1) (see also [18]). To be precise, the asymptotic expansion of the free resolvent for the three-dimensional case is

$$R_0(z) = \sum_{j=0}^{\infty} (iz^{1/2})^j G_j^{(3)}$$

where $G_j^{(3)}$ is the integral operator with the kernel $|\mathbf{x} - \mathbf{y}|^{j-1}/4\pi j!$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ [14]. Therefore the asymptote of $\exp(-itH_0)$ for the three-dimensional case becomes

$$\exp(-itH_0) = \pi^{3/2} (it)^{-3/2} \langle \varphi_0^{(3)}, \cdot \rangle \varphi_0^{(3)} + o(t^{-3/2}) \quad t \rightarrow \infty \quad (3.5)$$

in $\mathbf{B}(s, s')$, with large enough s and s' . Here $\varphi_0^{(3)}(\mathbf{x}) := (2\pi)^{-3/2}$ is the zero-energy eigenfunction of H_0 . Note that $\varphi_0^{(3)}$ does not yield the $t^{-1/2}$ -term in the expansion series, unlike in the one-dimensional case.

Formula (3.3) does not provide us with information at each point x . However, it is useful for calculating the quantities through the norm or the inner product, such as the survival probability. Suppose $s, s' > j+1/2, \sigma \in \mathbb{R}$, and $M \in \mathbf{B}(-s', \sigma)$. Then, since G_j is considered as a vector in $\mathbf{B}(s, -s')$, we see that $\|MG_j \psi\|_{\sigma}$ is well defined for all $\psi \in L^{2,s}(\mathbb{R})$. Therefore, for $s, s' > \max\{3n + 3/2, 5/2\}$ and $M \in \mathbf{B}(-s', \sigma)$, we have from (3.3)

$$\begin{aligned} & \left\| M \exp(-itH_0) \psi \right\|_{\sigma} - \pi^{-1} \left\| \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2) (it)^{-j-1/2} M G_{2j} \psi \right\|_{\sigma} \\ & \leq \left\| \exp(-itH_0) - \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2) (it)^{-j-1/2} G_{2j} \right\|_{s, -s'} \\ & = o(t^{-n-1/2}) \end{aligned} \quad (3.6)$$

for all $\psi \in L^{2,s}(\mathbb{R})$. For example, we can take $(1+x^2)^{s'/2}E(B)$ for M with $\sigma = -s'$, where $E(B)$ is the spectral measure of the position operator and B an arbitrary bounded interval of \mathbb{R} , i.e. $(E(B)\psi)(x) = \psi(x)$ ($x \in B$) or 0 ($x \notin B$). Then, $\|M \exp(-itH_0)\psi\|_{-s'}^2 = \|E(B) \exp(-itH_0)\psi\|^2 = \int_B |\psi(x, t)|^2 dx$, and the last quantity is called the nonescape probability, which is the probability of finding the particle in B at a time t . From a similar argument, we also have

$$\begin{aligned} & \left| \langle \phi, \exp(-itH_0)\psi \rangle - \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2) (it)^{-j-1/2} \langle \phi, G_{2j}\psi \rangle \right| \|\phi\|_s^{-1} \|\psi\|_s^{-1} \\ & \leq \left\| \exp(-itH_0) - \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2) (it)^{-j-1/2} G_{2j} \right\|_{s,-s'} \\ & = o(t^{-n-1/2}) \end{aligned} \tag{3.7}$$

for $s, s' > \max\{3n+3/2, 5/2\}$ with $s \geq s'$ and all $\phi, \psi \in L^{2,s}(\mathbb{R})$. The asymptotic formula for the survival amplitude of ψ is the special case of (3.7).

4. Dependence on the initial momentum distribution

In practical situations, it sometimes happens that there are no contributions from some of the G_{2j} to quantities such as $\|M \exp(-itH_0)\psi\|_\sigma$ or $\langle \phi, \exp(-itH_0)\psi \rangle$, when they act on a certain vector ψ . In this section, we confine ourselves to such situations for the survival amplitude $\langle \psi, \exp(-itH_0)\psi \rangle$.

Lemma 4.1. *Let $n = 0, 1, \dots$. If $s > n + 1/2$ and $\psi \in L^{2,s}(\mathbb{R})$, then $x^j \psi(x) \in L^1(\mathbb{R})$, for all $j = 0, 1, \dots, n$.*

Proof. The statement is obtained straightforwardly: $\int_{\mathbb{R}} |x^j \psi(x)| dx = \int_{\mathbb{R}} (1+x^2)^{-(s-j)/2} (1+x^2)^{(s-j)/2} |x^j \psi(x)| dx \leq \left[\int_{\mathbb{R}} (1+x^2)^{-(s-j)} dx \right]^{1/2} \left[\int_{\mathbb{R}} (1+x^2)^s |\psi(x)|^2 dx \right]^{1/2} < \infty$ for all $j = 0, 1, \dots, n$ with $s > n + 1/2$. \square

Lemma 4.2. *Let $m = 1, 2, \dots$. If $s > \max\{2(m-1), m\} + 1/2$ and $\psi \in L^{2,s}(\mathbb{R})$, then the following three statements are equivalent:*

- (a) $\hat{\psi}(k) = O(k^m), k \rightarrow 0$.
- (b) $\langle \psi, G_{2j}\psi \rangle = 0, j = 0, 1, \dots, m-1$.
- (c) $\int_{\mathbb{R}} x^j \psi(x) dx = 0, j = 0, 1, \dots, m-1$.

In particular, we have $G_{2j}\psi = 0, j = 0, 1, \dots, (m-1)/2$ for odd m , or $G_{2j}\psi = 0, j = 0, 1, \dots, m/2 - 1$ for even m .

Proof. Suppose that $\psi \in L^{2,s}(\mathbb{R})$ with $s > \max\{2(m-1), m\} + 1/2$. Then by lemma 4.1, we first see that $x^j \psi(x) \in L^1(\mathbb{R})$ for $j = 0, 1, \dots, \max\{2(m-1), m\}$. Since $e^{-ikx} = \sum_{j=0}^{m-1} (-ikx)^j / j! + (-ikx)^m \int_0^1 t^{m-1} e^{-ikx(1-t)} dt / (m-1)!$, the fact that (c) implies (a) immediately follows, by using

$$\begin{aligned} |\hat{\psi}(k)| &= \left| \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[\sum_{j=0}^{m-1} \frac{(-ikx)^j}{j!} + \frac{(-ikx)^m}{(m-1)!} \int_0^1 t^{m-1} e^{-ikx(1-t)} dt \right] \psi(x) dx \right| \\ &\leq \frac{|k|^m}{\sqrt{2\pi}(m-1)!} \int_{\mathbb{R}} |x^m \psi(x)| dx = O(k^m) \quad k \rightarrow 0. \end{aligned} \tag{4.1}$$

To prove the fact that (a) implies (c), let us remember that $\hat{\psi}(k) = O(k^m)$ ($k \rightarrow 0$) means that for some δ and some finite $C \geq 0$, $|\hat{\psi}(k)/k^m| \leq C$ for all k satisfying $0 < |k| < \delta$. Then, from (4.1), $\int_{\mathbb{R}} x^j \psi(x) dx$ should vanish for all $j = 0, 1, \dots, m$. The fact that (c) implies (b) follows straightforwardly from the identity

$$-2(2j)! \langle \psi, G_{2j} \psi \rangle = \sum_{i=0}^{2j} (-1)^{2j-i} \binom{2j}{i} \int_{\mathbb{R}} x^{2j-i} \overline{\hat{\psi}(x)} dx \int_{\mathbb{R}} y^i \psi(y) dy \quad (4.2)$$

where the bar ($\bar{}$) denotes the complex conjugate. To prove the fact that (b) implies (c), we first use the assumption that $-2\langle \psi, G_0 \psi \rangle = |\int_{\mathbb{R}} \psi(x) dx|^2 = 0$. Then from (4.2) for $j = 1$ we have $|\int_{\mathbb{R}} x \psi(x) dx|^2 = 0$, from which the remaining equalities recursively follow. For the proof of the last part of the lemma, we note that for $s, s' > 2j + 1/2$ and $\psi \in L^{2,s}(\mathbb{R})$,

$$G_{2j} \psi = 0 \Leftrightarrow \|G_{2j} \psi\|_{-s'} = 0 \Leftrightarrow \int_{\mathbb{R}} x^i \psi(x) dx = 0 \quad (i = 0, 1, \dots, 2j).$$

Hence, if (c) holds, we obtain the equality $G_{2j} \psi = 0$ for all j satisfying $2j \leq m - 1$. This completes the proof. \square

Now we shall derive the asymptotic formula for the survival amplitude by combining theorem 3.1 with lemma 4.2. The asymptotic formula itself immediately follows from (3.7) with $\phi = \psi$, under the assumption in theorem 3.1. We also see that $\max\{3n + 3/2, 5/2\} \geq 3n + 3/2 \geq \max\{2(m - 1), m\} + 1/2$ for $m \leq n$, and thus the assumption in lemma 4.2 is included in theorem 3.1. Hence, we finally obtain the following theorem for the survival amplitude of ψ , which is closely connected to the behaviour of ψ at zero momentum.

Theorem 4.3. *Let $m, n, m \leq n$ be non-negative integers, $s > \max\{3n + 3/2, 5/2\}$, and $\psi \in L^{2,s}(\mathbb{R})$. Then it follows that*

$$\langle \psi, \exp(-itH_0)\psi \rangle = \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \Gamma(j + 1/2) (it)^{-j-1/2} \langle \psi, G_{2j} \psi \rangle + o(t^{-n-1/2}). \quad (4.3)$$

In particular, for some $m \geq 1$, if $\hat{\psi}(k) = O(k^m)$ as $k \rightarrow 0$, then $\langle \psi, G_{2j} \psi \rangle = 0$ for all $j = 0, 1, \dots, m - 1$, and vice versa.

The asymptotic formula (4.3) can also be written without using G_{2j} . Let $n = 0, 1, \dots$. If $s > n + 1/2$ and $\psi \in L^{2,s}(\mathbb{R})$, $\hat{\psi}(k)$ is n -times continuously differentiable in k with $\hat{\psi}^{(n)}(k) = (-i)^n (2\pi)^{-1/2} \int_{\mathbb{R}} x^n e^{-ikx} \psi(x) dx$. Then, we have from lemma 4.2 and equation (4.2) that if $\hat{\psi}(k) = O(k^m)$ as $k \rightarrow 0$,

$$\langle \psi, G_{2m} \psi \rangle = -\frac{(-1)^m}{2(2m)!} \binom{2m}{m} \left| \int_{\mathbb{R}} x^m \psi(x) dx \right|^2 = -\frac{(-1)^m \pi}{(m!)^2} |\hat{\psi}^{(m)}(0)|^2. \quad (4.4)$$

This expression also holds for $m = 0$ (see (3.4)). Therefore, we obtain from (4.3) the asymptotic formula for the survival probability,

$$|\langle \psi, \exp(-itH_0)\psi \rangle|^2 = t^{-2m-1} \frac{\Gamma(m + 1/2)^2}{(m!)^4} |\hat{\psi}^{(m)}(0)|^4 + o(t^{-2m-1}). \quad (4.5)$$

In theorem 4.3, the assumption that $\psi \in L^{2,s}(\mathbb{R})$ with sufficiently large s is technically required. According to expression (4.5), it is worth reviewing this assumption in the momentum representation. Let $\psi \in L^{2,s}(\mathbb{R})$ ($s \geq 0$) and $[s]$ denote the smallest integer less than or equal to s . Then, by the Plancherel theorem, we have

$$\infty > \int_{\mathbb{R}} (1 + x^2)^n |\psi(x)|^2 dx \geq \int_{\mathbb{R}} |x^n \psi(x)|^2 dx = \int_{\mathbb{R}_k} |\hat{\psi}^{(n)}(k)|^2 dk$$

for all $n = 0, 1, \dots, [s]$. It should be noted here that $\hat{\psi}(k)$ is implicitly guaranteed to be $[s]$ -times differentiable. Hence, as an obvious case, we can find the following subspace,

$$\mathcal{D} := \{\hat{\psi} \in C^\infty(\mathbb{R}_k) \mid \hat{\psi}^{(n)} \in L^2(\mathbb{R}_k), n = 0, 1, \dots\} \tag{4.6}$$

which satisfies $\mathcal{D} \subset \{\hat{\psi} \in L^2(\mathbb{R}_k) \mid \psi \in L^{2,s}(\mathbb{R})\}$ for all $s \geq 0$. Then, equations (3.6) and (3.7) with an arbitrary n can be applied to the wavefunctions belonging to the above \mathcal{D} . Examples of such (initial) wavefunctions include $k^l/(1+k^{2m})^\alpha$ and $k^n \exp(-k^2)$, where $l, m, n = 0, 1, \dots$ and $\alpha > 0$ with $2m\alpha - l > 1/2$.

In order to see some implications of theorem 4.3, let us refer to the following two examples. We first consider the rapidly decreasing functions, $\hat{\phi}_m(k) = N_m k^m \exp(-a_0 k^2)$, as initial wavefunctions, where $m = 0, 1, \dots, a_0 > 0$, and $N_m := [\Gamma(m + 1/2)/(2a_0)^{m+1/2}]^{-1/2}$ are the normalization constants. Then it is obtained through the Laplace transform that

$$\begin{aligned} \langle \phi_m, \exp(-itH_0)\phi_m \rangle &= \int_{-\infty}^{\infty} |\hat{\phi}_m(k)|^2 \exp(-itk^2) dk \\ &= [1 + it/(2a_0)]^{-(m+1/2)} = (it/2a_0)^{-(m+1/2)} [1 + O(t^{-1})]. \end{aligned} \tag{4.7}$$

On the other hand, we see that the right-hand side of (4.5) exactly corresponds to that of (4.7) in the leading order. The other example is a special case that $\hat{\psi}^{(n)}(0) = 0$ for all $n = 0, 1, \dots$. It is worth noting that for such an initial wavefunction, we clearly see from (4.5) that

$$|\langle \psi, \exp(-itH_0)\psi \rangle|^2 = o(t^{-2n-1})$$

for every $n \geq 0$; that is the survival probability decays faster than any power of t^{-1} . However, it must decay slower than any exponential at long times for $H_0 \geq 0$ [1]. This strange decay behaviour is also found in a study of the time operator (proposition 3.2 in [12]). The set of such a special wavefunction is given, e.g., by $C_i := \{\hat{\psi} \in C_0^\infty(\mathbb{R}_k) \mid \exists k_0 > 0; \hat{\psi}(k) = 0, \text{ for } k \in [-k_0, k_0]\}$. We see from (4.6) that $C_i \subset \mathcal{D}$. A wavefunction in C_i has a positive lower-bound k_0^2 on energy. For instance, the following function,

$$\hat{\psi}(k) = \begin{cases} \exp(-1/[k_0^2 - (k-d)^2]) & (|k-d| < k_0) \\ 0 & (|k-d| \geq k_0) \end{cases}$$

where $d > k_0 > 0$ surely belongs to C_i .

5. Concluding remarks

We have derived the asymptotic expansion of the time evolution operator for the one-dimensional free particle system, in terms of the operators which are expansion coefficients of the free resolvent at small energies. This enables us to obtain the asymptotic formula for the survival probability of ψ , and also to evaluate, in a systematic way, the condition for the initial wavefunction ψ which makes the first several terms of the asymptotic formula vanish. We have found that if $\hat{\psi}(k) = O(k^m)$ for some non-negative integer m at zero momentum, the asymptotic power of t^{-1} for the survival probability must be $2m + 1$. In other words, the information about the initial momentum distribution in the vicinity of zero momentum is reflected in the asymptotic decay form t^{-2m-1} at long times. Our results are essentially due to the choice of the initial wavefunctions ψ in $L^{2,s}(\mathbb{R})$ with sufficiently large s . This guarantees the existence of the higher derivatives at zero momentum (see the comment after equation (4.5)). However, there is another wavefunction such that $\hat{\psi}^{(n)}(0) = 0$ up to $n = m - 1$, while its m th derivative $\hat{\psi}^{(m)}(0)$ diverges. Related wavefunctions are considered in [8–11]. For such states, the asymptotic formula (4.5) is not correct. Indeed, the actual asymptotic decay form of the survival probability includes terms of non-odd powers of t^{-1} . We hope to address this issue in future.

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Appendix A

Dependence of the survival probability on the initial state ψ for a one-dimensional free particle system can be seen in the following inequality (theorem 4.1 and section VI in [12]),

$$|\langle \psi, \exp(-itH_0)\psi \rangle|^2 \leq \frac{4\|T_0\psi\|^2\|\psi\|^2}{t^2} \quad t \in \mathbb{R} \quad (\text{A1})$$

where T_0 is the Aharonov–Bohm time operator [19]. This provides us with an interpretation of $\|T_0\psi\|$, or of the time uncertainty calculated from T_0 . The relevant information herein is that for any L^2 -function ψ whose several moments are finite, e.g. $\psi \in L^{2,s}(\mathbb{R})$ ($s > 3/2$), we have

$$\|T_0\psi\| < \infty \iff \hat{\psi}(k) = O(k^2) \quad k \rightarrow 0. \quad (\text{A2})$$

This implies, together with (A1), that condition (A2) at zero momentum imposes on the survival amplitude $\langle \psi, \exp(-itH_0)\psi \rangle$ a decay faster than t^{-1} , which is obviously faster than $t^{-1/2}$ of the usual decay law for the one-dimensional free particle system.

To prove relation (A2), we first define the Aharonov–Bohm time operator T_0 , which is mathematically well treated in the scheme of the axiomatic quantum mechanics [12, 20]. We define this operator as follows: the domain of T_0 is

$$D(T_0) := \left\{ \psi \in L^2(\mathbb{R}) \mid \lim_{k \rightarrow 0} \hat{\psi}(k)/|k|^{1/2} = 0 \text{ and } (\widehat{T_0\psi})(k) \in L^2(\mathbb{R}_k) \right\} \quad (\text{A3})$$

and its action

$$(\widehat{T_0\psi})(k) = \frac{i}{4} \left(\frac{d\hat{\psi}(k)/k}{dk} + \frac{1}{k} \frac{d\hat{\psi}(k)}{dk} \right) \quad \text{a.e.} \quad k \in \mathbb{R}_k \quad \psi \in D(T_0) \quad (\text{A4})$$

where $\hat{\psi}(k)$ is assumed to be differentiable everywhere except at the origin. For a ψ that belongs to $L^{2,s}(\mathbb{R})$ ($s > 3/2$), $\psi(x)$ and $x\psi(x)$ are in $L^1(\mathbb{R})$ by lemma 4.1. This implies that $\hat{\psi}(k)$ is differentiable everywhere including the origin. Then, it follows as in (4.1) that for $k \rightarrow 0$

$$\hat{\psi}(k) = \hat{\psi}(0) + k\hat{\psi}'(0) + O(k^2) \quad \hat{\psi}'(k) = \hat{\psi}'(0) + O(k). \quad (\text{A5})$$

Hence

$$(\widehat{T_0\psi})(k) = \frac{i}{4} \left(-\frac{\hat{\psi}(0)}{k^2} + \frac{\hat{\psi}'(0)}{k} \right) + O(1) \quad (\text{A6})$$

for small k . Furthermore, since $\psi \in L^{2,s}(\mathbb{R}) \subset L^{2,1}(\mathbb{R})$, $\int_{\mathbb{R}} |x\psi(x)|^2 dx = \int_{\mathbb{R}_k} |\hat{\psi}'(k)|^2 dk < \infty$. Thus, $(\widehat{T_0\psi})(k)$ is assured to be square integrable on $(-\infty, -\delta] \cup [\delta, \infty)$ for an arbitrary $\delta > 0$.

Now, to prove the fact that $\|T_0\psi\| < \infty$ implies $\hat{\psi}(k) = O(k^2)$, $k \rightarrow 0$ in relation (A2), let us suppose that $(\widehat{T_0\psi})(k) \in L^2(\mathbb{R}_k)$. Then, from the general property of L^2 -functions, we see that $(\widehat{T_0\psi})(k) \in L^1([-\delta, \delta])$. This contradicts (A6) unless $\hat{\psi}(0) = \hat{\psi}'(0) = 0$. Thus we have that $\hat{\psi}(k) = O(k^2)$. Conversely, if $\hat{\psi}(k) = O(k^2)$, it follows from (A5) that $\hat{\psi}(0) = \hat{\psi}'(0) = 0$. Then, (A6) implies that $(\widehat{T_0\psi})(k) \in L^2([-\delta, \delta])$. Hence, $(\widehat{T_0\psi})(k)$ belongs to $L^2(\mathbb{R}_k)$, and the proof of relation (A2) is completed.

Appendix B

In this appendix, we derive formula (3.2) which directly relates $\exp(-itH_0)$ to the spectral density. Let us remember that the time evolution operator for a one-dimensional free particle system is explicitly represented as

$$\psi(x, t) = (\exp(-itH_0)\psi)(x) = (4\pi it)^{-1/2} \int_{\mathbb{R}} \exp(i|x - y|^2/4t)\psi(y) dy = O(t^{-1/2}) \quad (\text{B1})$$

for all $\psi \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $t \neq 0$ [17]. Note that $L^{2,s}(\mathbb{R}) \subset L^1(\mathbb{R})$ for $s > 1/2$. Then, $\exp(-itH_0)$ is considered as an integral operator belonging to $\mathbf{B}(s, -s')$ for $s, s' > 1/2$.

Let $s, s' > 1/2$ and $\psi \in L^{2,s}(\mathbb{R})$. We have an equality

$$\left(\int_r^R E'(\lambda) e^{-it\lambda} d\lambda \psi \right) (x) = \int_{\mathbb{R}} dy \left[\int_r^R \frac{\cos(\lambda^{1/2}|x - y|) e^{-it\lambda}}{2\pi\lambda^{1/2}} d\lambda \right] \psi(y) \quad \text{a.e. } x \in \mathbb{R}$$

for positive r and R . Then it follows from (B1) that

$$\begin{aligned} \left\| \int_r^R E'(\lambda) e^{-it\lambda} d\lambda \psi - \exp(-itH_0)\psi \right\|_{-s'}^2 &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_r^R \frac{\cos(\lambda^{1/2}|x - y|) e^{-it\lambda}}{2\pi\lambda^{1/2}} d\lambda \right. \\ &\quad \left. - \frac{\exp(-|x - y|^2/4it)}{(4\pi it)^{1/2}} \right|^2 \frac{dx dy}{(1 + x^2)^{s'}(1 + y^2)^s}. \end{aligned} \quad (\text{B2})$$

We here see that

$$\sup_{x \in \mathbb{R}, R > 0} \left| \int_0^R \frac{\cos(\lambda^{1/2}|x|) e^{-it\lambda}}{2\pi\lambda^{1/2}} d\lambda \right| < \infty. \quad (\text{B3})$$

To derive this, we have assumed $t > 0$; however, the following argument can be applied to negative t ,

$$\begin{aligned} \left| \int_0^R \frac{\cos(\lambda^{1/2}|x|) e^{-it\lambda}}{2\pi\lambda^{1/2}} d\lambda \right| &= (2\pi)^{-1} \left| \int_{-R^{1/2}}^{R^{1/2}} \exp(-it(\xi + a)^2) d\xi \right| \\ &= (2\pi)^{-1} \left| \left[\int_0^{R^{1/2}+a} \exp(-it\xi^2) d\xi + \int_{-R^{1/2}+a}^0 \exp(-it\xi^2) d\xi \right] \right| \end{aligned}$$

where $a = |x|/2t$. The last two integrals are estimated, by the complex integrals with the contour in the fourth quadrant, to be

$$\begin{aligned} \left| \int_0^{|R^{1/2} \pm a|} \exp(-it\xi^2) d\xi \right| &\leq \frac{\pi(1 - \exp(-t|R^{1/2} \pm a|^2))}{4t|R^{1/2} \pm a|} + \int_0^{|R^{1/2} \pm a|} \exp(-tr^2) dr \\ &\leq \frac{\pi}{4t} \sup_{x > 0} \frac{1 - \exp(-tx^2)}{x} + \frac{1}{2} \sqrt{\frac{\pi}{t}} < \infty. \end{aligned}$$

This leads to (B3). We also see straightforwardly

$$\lim_{R \rightarrow \infty} \lim_{r \downarrow 0} \int_r^R \frac{\cos(\lambda^{1/2}|x - y|) e^{-it\lambda}}{2\pi\lambda^{1/2}} d\lambda = \frac{\exp(-|x - y|^2/4it)}{(4\pi it)^{1/2}}. \quad (\text{B4})$$

Therefore, by the dominated convergence theorem, we have from (B2), (B3) and (B4)

$$\sup_{\psi \in L^{2,s}(\mathbb{R}), \psi \neq 0} \left\| \exp(-itH_0)\psi - \int_r^R E'(\lambda) e^{-it\lambda} d\lambda \psi \right\|_{-s'}^2 \|\psi\|_s^{-2} \rightarrow 0$$

as $r \downarrow 0$ and $R \rightarrow \infty$, and we finally obtain formula (3.2).

Appendix C

In order to prove theorem 3.1, we first summarize the several properties of $E'(\lambda)$. By lemma 2.3, we see that for $n \geq 0$ the remainder $F_n(\lambda)$ in (3.1) is $(n+1)$ -times differentiable, in $\mathbf{B}(s, -s')$ for $s, s' > 2n+(n+1)+1/2 = 3n+3/2$, and satisfies that $(d/d\lambda)^r F_n(\lambda) = o(\lambda^{n-r-1/2})$ as $\lambda \downarrow 0$ ($r = 0, 1, \dots, n+1$). On the other hand, we see from lemma 2.4 that $(d/d\lambda)^r E'(\lambda) = O(\lambda^{-(r+1)/2})$ as $\lambda \rightarrow \infty$, in $\mathbf{B}(s, -s')$ for $s, s' > (m+1)+1/2 = m+3/2$ ($r = 0, 1, \dots, m+1$). In particular, if $m \geq 1$, $(d/d\lambda)^{m+1} E'(\lambda)$ is integrable on $[\delta, \infty)$ for an arbitrary $\delta > 0$.

Let us now split the integral in (3.2) into two parts by writing

$$E'(\lambda) = \phi(\lambda)E'(\lambda) + (1 - \phi(\lambda))E'(\lambda)$$

where $\phi \in C_0^\infty([0, \infty))$ and satisfies $\phi(\lambda) = 1$ in a neighbourhood of $\lambda = 0$. Such a function is realized by $f(\lambda) = 1 - \int_0^\lambda g(x) dx$, where $g(x) = h(x)/\int_{\mathbb{R}} h(x) dx$ and $h(x) = \exp(-1/[1 - (x-d)^2])$ ($|x-d| < 1$) or 0 ($|x-d| \geq 1$) with $d > 1$.

From lemma 10.1 in [14] and the discussion as mentioned above, we see that $(1 - \phi(\lambda))E'(\lambda)$ has a contribution of $o(t^{-m-1})$ to $\exp(-itH_0)$ in $\mathbf{B}(s, -s')$, where $m \geq 1$ and $s, s' > m + 3/2$.

On the other hand, the contribution of $\phi(\lambda)E'(\lambda)$ to $\exp(-itH_0)$ gives the main part of the asymptotic expansion. Then, the coefficient of G_{2j} is given by

$$\begin{aligned} \int_0^\infty \phi(\lambda)\lambda^{j-1/2} e^{-it\lambda} d\lambda &= i^j \frac{d^j}{dt^j} \left[\int_0^\infty \lambda^{-1/2} e^{-it\lambda} d\lambda + \int_0^\infty (\phi(\lambda) - 1)\lambda^{-1/2} e^{-it\lambda} d\lambda \right] \\ &= \Gamma(j+1/2)(it)^{-j-1/2} + i^j \frac{d^j}{dt^j} \int_0^\infty (\phi(\lambda) - 1)\lambda^{-1/2} e^{-it\lambda} d\lambda. \end{aligned} \quad (\text{C1})$$

Note that since $\phi(\lambda) - 1$ and all its derivatives vanish in the neighbourhood of $\lambda = 0$ and $\phi \in C_0^\infty([0, \infty))$, the last term in (C1) decays faster than any negative power of t . Furthermore, we understand, from lemma 10.2 in [14] and the discussion in appendix A, that if $s, s' > 3n + 3/2$, any contribution of the Fourier transform of the remainder $\phi(\lambda)F_n(\lambda)$ to $\exp(-itH_0)$ is of $o(t^{-n-1/2})$ in the norm of $\mathbf{B}(s, -s')$. Summarizing the above arguments, we finally obtain, under the condition $s, s' > \max\{3n + 3/2, 5/2\}$,

$$\begin{aligned} &\left\| \exp(-itH_0) - \pi^{-1} \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2)(it)^{-j-1/2} G_{2j} \right\|_{s, -s'} \\ &\leq \left\| \exp(-itH_0) - \int_r^R E'(\lambda) e^{-it\lambda} d\lambda \right\|_{s, -s'} + \left\| \int_r^R E'(\lambda) e^{-it\lambda} d\lambda \right. \\ &\quad \left. - \sum_{j=0}^n (-1)^{j-1} \Gamma(j+1/2)(it)^{-j-1/2} G_{2j} \right\|_{s, -s'} \rightarrow 0 + o(t^{-n-1/2}) \end{aligned}$$

as $r \downarrow 0$ and $R \rightarrow \infty$. This is just the asymptotic expansion of $\exp(-itH_0)$ in (3.3).

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